IV. Egyptian Mathematics

There are five main features of Egyptian mathematics on which we will focus. We will experience them first hand in homework II and in worksheet #3.

- The absence of place-value notation in their Hieroglyphic system of writing numbers
- Their advances in geometry
- Their use of addition in place of multiplication
- The use of only unit fractions with attendant algorithms
- The use of base ten arithmetic

The first pharaoh to unite the upper and lower kingdoms of Egypt was the ruler of the southern, or upper kingdom, Narmer. He may have been the father of the king Aha. He united the two kingdoms in about 3100 B.C.E. This military victory is documented for us in the Narmer palette. The early pharaohs were buried in the ancient city of Abydos, in upper Egypt. There are inscriptions on ivory and wood in these pre-pyramidal tombs known as mastabas, which contain hieroglyphic numbers. Abydos predates Narmer, and this form of writing numerals goes back to the fifth millennium B.C.E. Abydos is about half way between Cairo and the Aswan dam.

There is a map of ancient Egypt at the end of this section on Egyptian mathematics.

There is a recently excavated area of older tombs in Abydos that lend credence to idea that the unification process took a few centuries to achieve. The king who finally united the upper and lower kingdoms may then have claimed full credit for himself. Older pharaohs known as Crocodile, Scorpion I and II, Iryhor and Ka have been identified, predating the first dynasty of Narmer.

The lack of a place-value notation and the use of complicated ideographic hieroglyphics severely hampered the development of calculation in Egypt. Here are slightly fanciful depictions of the base ten hieroglyphic numbers. There is no zero. There are no negative numbers.

The first row on this page shows their numbers 1, 3 10, 100 and 1,000.

The numbers 2 through 9 are appropriate bunches of lines. The numbers 11 through 19 just place the lines by the loaf of bread (?) symbol for 10. Similarly for 101, 1,001, using the coil of rope for 100 and the lotus plant for 1,000.
Below we have their symbols for 10,000, 100,000 and 1,000,000. The frog is usually more of a tadpole.

Since there was no place value 123 could be written several ways:

\[
\begin{align*}
\text{III} & \quad \text{III} \\
\text{en} & \quad \text{en}
\end{align*}
\]

and so on.

The ancient Egyptians were more explicitly interested in geometry than their Mesopotamian counterparts. This is perhaps due to the annual flooding of the Nile River which was a powerful influence on the entire society. The upper kingdom of Egypt is the Valley of the Nile, never more than 15 miles wide, and surrounded by limestone cliffs with the desert beyond. The boundaries of land worked by farmers had to be reset every year. The areas of such geometric figures as rectangles, trapezoids and triangles were well known to these people. They used the formula

\[ A = \frac{8d^2}{9} \]

for the area of a circle of diameter \( d \). This translates in our terms into

\[ \Pi = 3.16049 \]

This is not as accurate as the Babylonian value for \( \Pi \). Notice that both societies used such values for \( \Pi \) in practical ways, and seem to have had no interest in fantastic accuracy, or conceptual differences between numbers that are integers, and numbers like \( \Pi \).

The Egyptians of this ancient era were familiar with the use of similar figures in geometry.

The construction of large tombs such as the pyramids required huge organization of logistics, and excellent geometric engineering feats. We still are not sure how they accomplished what they did with the technology at their disposal. The great pyramid of Khufu (Greek Cheops) was completed in 2744 B.C.E. and is an astounding building.

In connection with these pyramids, the Egyptians found the formula

\[ V = \frac{h}{3}(a^2 + ab + b^2) \]

for the volume of a truncated, square based pyramid of height \( h \), with base \( a \) and upper square side \( b \). Again, we do not know how they did this. We would use calculus in some part of our derivation of this fact.

The ancient Egyptians did not multiply but used doubling, a simple form of addition to accomplish any integer multiplication. Here is an example.

To multiply 56 times 73 you take the larger number, 73. Line it up with the number 1:
You can get any integer by adding powers of 2.

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<td>4</td>
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<td>584</td>
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<td>16</td>
<td>1168</td>
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<td>32</td>
<td>2336</td>
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56 times 73 is gotten by adding 2336 + 1168 + 584 = 4088.

The ancient Egyptians used unit fractions (fractions with a numerator of 1) with the sole exception of 2/3. This is a less comfortable way to deal with fractions than the Mesopotamian unified place-value notation. They had extensive tables for reducing fractions with numerators larger than 1 to unit fractions. We find tables of fractions as well as problems involving them in the Rhind papyrus written about 1650 B.C.E by the scribe Ahmose. Egyptian writing, when not on stone monuments, ivory or wood, was on papyrus which was used as far back as the first dynasty, around 3100 B.C.E. This material rots in humid conditions, so the documentation in Egypt is less than in Mesopotamia.

In the Rhind Papyrus we find Ahmose using the splitting algorithm to create tables of unit fractions equivalent to other fractions:

\[
\frac{2}{n} = \frac{2}{n+1} + \frac{2}{n(n+1)}, \quad \text{where we assume } n \text{ is an odd number.}
\]

If \( n \) is even, cancel the 2 to get a unit fraction. The Egyptians also used

\[
\frac{2}{3} = \frac{1}{2} + \frac{1}{6}, \quad \text{a special case of the splitting algorithm}
\]

We are now in a position to examine Egyptian “division.” Again, they used doubling and halving to accomplish division. Suppose you want to know how many times 8 goes into 59.

You begin by doubling the divisor until the next double will exceed the dividend.

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<td>32</td>
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So add as many multiples of 8 as you can, 32 + 16 + 8 = 56, staying less than or equal to the dividend, in this case 59. We have 7 = 1 + 2 + 4 multiples of 8 here. We have 3 left over.

We now need to get 3 from 8. Start halving the 8.

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<tr>
<td>1/2</td>
<td>4</td>
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1/4 --- 2
1/8 --- 1, and notice that 1 + 2 = 3.

So $59/8 = 7 + 1/4 + 1/8$, the answer always being in unit fractions.

Another example is 83 divided by 7:

$$\begin{array}{cccc}
1 & \quad 7 \\
2 & \quad 14 \\
4 & \quad 28 \\
8 & \quad 56 \\
\hline
\text{so the whole number part of the answer is } & 11 = 1 + 2 = 8.
\end{array}$$

We have 6 left over. The halving is awful so we use the splitting algorithm instead:

$$\frac{6}{7} = \frac{2}{7} + \frac{2}{7} + \frac{2}{7} = \frac{1}{4} + \frac{1}{28} + \frac{1}{4} + \frac{1}{28} + \frac{1}{4} + \frac{1}{28} =$$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{14} + \frac{1}{28}$$

So the answer is $11 + \frac{1}{2} + \frac{1}{4} + \frac{1}{14} + \frac{1}{28}$.

We engage with worksheet #3, Ancient Egyptian Calculations, at this point.

The ancient Egyptians seem to have had a more limited ability to solve quadratic equations than the Mesopotamians, although they did some solving of these equations.

They also had less observational astronomy, though they had a 365 day year by 2776 B.C.E. Their new year was measured by the heliacal (rises at dawn and disappears in the sunlight) rising of the star we call Sirius, the dog star. To them it was the soul of Isis. Isis is the sister and wife of Osiris, both being the children of Geb (the earth) and Nut (the sky), and the parents of the god Horus, the falcon god. The Egyptians had a 12 month year with the day divided into 12 decans. These decans were longer or shorter, depending on the season of the year. It is not until Hipparchus around 175 B.C.E. proposed equal length hours that some educated people began using them. Popular use had to wait for the invention of mechanical clocks in the 14th century.

Homework II

1. Use the splitting algorithm to exhibit the following as sums of unit fractions:
   a. $2/7$
   b. $2/11$
   c. $2/13$
   d. $2/17$

2. Exhibit the fraction $2/97$ in two different ways as the sum of unit fractions.
3. Use the Egyptian method of doubling to calculate the following multiplications:
   a. 23x134  
   b. 14x58  
   c. 31x192

4. Use the Egyptian method of doubling and halving to calculate the following divisions:
   a. 1060 divided by 12  
   b. 19 divided by 8.

5. The Moscow Papyrus has the formula mentioned above for the volume of a truncated square pyramid.
   a. Suppose a = 3, b = 7 and h = 6. What is V?
   b. Suppose V = 183, a = 5 and b = 4, what is h?

6. Find the lengths of the sides of a rectangle whose area is 20 square feet and whose width is 4/5 of its length.

7. (Math Credit) Suppose you know that the volume of a square pyramid is
   \[ V = \frac{1}{3} \cdot a^2h \]  
   where a is the side of the base and h is the height.
   Use this extra fact to prove the Egyptian formula for the volume of a truncated square pyramid.

Here is a brief timeline of the history that parallels the mathematical developments described above.

- Before 3100 B.C.E. there were two kingdoms in Egypt already, one in the upper, or southern area, in the narrow Nile valley, and one in the northern, wide delta of the Nile. The city of Memphis was a dividing point. A thriving culture was mostly beyond our knowledge there. But there is evidence that the isolated cultures at several settlements along the Nile River were swallowed by the more advanced culture at Nagada. This culture spread both downriver and upriver around 4,000 B.C.E.

- Around 3100 Narmer united the two, establishing the first Dynasty, which lasted from 3100 to 2890 B.C.E. It may be the case that Narmer chose to claim total credit for the unification, while the actual process had been going on for several centuries. A second dynasty extended from 2890 to 2686 B.C.E.

- The Old Kingdom (4th to 8th dynasties) is usually dated from about 2575 to 2181 B.C.E. when the culture developed more of its science and mathematics and built its most impressive structures. During this period the religion came into being and
The pharaoh was equated with god. The **capital city of Memphis** was established in this era.

- The **first intermediate period** (9\textsuperscript{th} to 11\textsuperscript{th} dynasties) from 2125 to 1975 B.C.E. had Egypt as two kingdoms again. The capitals were at **Memphis** in the north and **Thebes** in the south.

- The **Middle Kingdom** (11\textsuperscript{th} to 14\textsuperscript{th} dynasties) extends from 1975 to 1640 B.C.E. **Mentuhotep** reunited all of Egypt. During this period the capital was moved upriver from Memphis to Thebes. This was the classical period for arts and literature.

- The **second intermediate period** (15\textsuperscript{th} to 17\textsuperscript{th} dynasties) from 1630 to 1520 B.C.E. saw the **Hyksos** conquer the northern kingdom. The Hyksos had some Canaanite gods and were from the east. They introduced horse-drawn chariots into Egypt. **Ahmose I** drove the Hyksos out of Egypt.

- The New kingdom (18\textsuperscript{th} to 20\textsuperscript{th} dynasties) extends from 1539 to 1075 B.C.E. and saw a woman as pharaoh, **Hatshepsut**. The pharaoh **Akhenaten** introduced monotheism, but it did not last. **Nefertiti** was his queen. Elaborate tombs were built in the Valley of the Kings. **Tutankhamun** was the pharaoh from 1334 to 1325 B.C.E. **Ramses II**, the most powerful pharaoh, ruled from 1279 to 1213 B.C.E.

- **Nubia** conquered Egypt around 710 B.C.E.

- The **Persians** conquered Egypt in 525 B.C.E.

- In 332 B.C.E. **Alexander the Great** conquered Egypt.